Exercise 23

Solve the differential equation using the method of variation of parameters.

$$y'' + y = \sec^2 x, \quad 0 < x < \pi/2$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$r^2 e^{rx} + e^{rx} = 0$$

Divide both sides by e^{rx}.

$$r^2 + 1 = 0$$

Solve for r.

$$r = \{-i, i\}$$

Two solutions to the ODE are e^{-ix} and e^{ix}. By the principle of superposition, then,

$$\begin{aligned} y_c(x) &= C_1 e^{-ix} + C_2 e^{ix} \\ &= C_1(\cos x - i \sin x) + C_2(\cos x + i \sin x) \\ &= (C_1 + C_2)\cos x + (-iC_1 + iC_2)\sin x \\ &= C_3 \cos x + C_4 \sin x. \end{aligned}$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' + y_p = \sec^2 x \tag{2}$$

In order to obtain a particular solution, use the method of variation of parameters: Allow the parameters in the complementary solution to vary.

$$y_p = C_3(x)\cos x + C_4(x)\sin x$$

Differentiate it with respect to x.

$$y_p' = C_3'(x)\cos x + C_4'(x)\sin x - C_3(x)\sin x + C_4(x)\cos x$$

If we set

$$C_3'(x)\cos x + C_4'(x)\sin x = 0, \tag{3}$$

then

$$y_p' = -C_3(x)\sin x + C_4(x)\cos x.$$

Differentiate it with respect to x once more.

$$y_p'' = -C_3'(x)\sin x + C_4'(x)\cos x - C_3(x)\cos x - C_4(x)\sin x$$

Substitute these formulas into equation (2).

$$\left[-C_3'(x)\sin x + C_4'(x)\cos x - \cancel{C_3(x)\cos x} - \cancel{C_4(x)\sin x}\right] + \left[\cancel{C_3(x)\cos x} + \cancel{C_4(x)\sin x}\right] = \sec^2 x$$

Simplify the result.

$$-C_3'(x)\sin x + C_4'(x)\cos x = \sec^2 x \tag{4}$$

Multiply both sides of equation (3) by $\sin x$, and multiply both sides of equation (4) by $\cos x$.

$$C_3'(x)\cos x\sin x + C_4'(x)\sin^2 x = 0$$

$$-C_3'(x)\cos x\sin x + C_4'(x)\cos^2 x = \sec x$$

Add the respective sides of these equations to eliminate $C_3'(x)$.

$$C_4'(x) = \sec x$$

Integrate this result to get $C_4(x)$, setting the integration constant to zero.

$$C_4(x) = \ln|\sec x + \tan x|$$

The absolute value can be dropped since $0 < x < \pi/2$. Multiply both sides of equation (3) by $\cos x$, and multiply both sides of equation (4) by $-\sin x$.

$$C_3'(x)\cos^2 x + C_4'(x)\cos x\sin x = 0$$

$$C_3'(x)\sin^2 x - C_4'(x)\cos x\sin x = -\sec^2 x\sin x$$

Add the respective sides of these equations to eliminate $C_4'(x)$.

$$C_3'(x) = -\sec^2 x\sin x$$

Integrate this result to get $C_3(x)$, setting the integration constant to zero.

$$\begin{aligned} C_3(x) &= \int^x C_3'(w)\,dw \\ &= -\int^x \frac{\sin w}{\cos^2 w}\,dw \\ &= \int^{\cos x} \frac{du}{u^2} \\ &= \left(-\frac{1}{u}\right)\Big|^{\cos x} \\ &= -\frac{1}{\cos x} \end{aligned}$$

Therefore,

$$\begin{aligned} y_p &= C_3(x)\cos x + C_4(x)\sin x \\ &= \left(-\frac{1}{\cos x}\right)\cos x + \ln(\sec x + \tan x)\sin x \\ &= -1 + \ln(\sec x + \tan x)\sin x, \end{aligned}$$

and the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_3\cos x + C_4\sin x - 1 + \ln(\sec x + \tan x)\sin x, \end{aligned}$$

where C_3 and C_4 are arbitrary constants.